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AUTHOR(S):

Mizuta, Yoshihiro

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Growth properties for generalized Riesz potentials in central Herz-Morrey spaces

Yoshihiro Mizuta

Abstract

Riesz decomposition theorem says that a superharmonic function on the punctured unit ball B_0 is represented as the sum of a generalized potential and a harmonic function outside the origin. Our first aim in this note is to study growth properties near the origin for generalized Riesz potentials of functions in central Herz-Morrey spaces on B_0 .

We know another Riesz decomposition theorem which says that a superharmonic function on the unit ball B is represented as the sum of another generalized potential and a harmonic function on B . Our second aim in this note is to obtain growth properties near the boundary ∂B for generalized Riesz potentials of functions in central Herz-Morrey spaces on B .

A continuous function u on an open set Ω is called monotone in the sense of Lebesgue [18] if for every relatively compact open set $G \subset \Omega$,

$$\max_{\overline{G}} u = \max_{\partial G} u \quad \text{and} \quad \min_{\overline{G}} u = \min_{\partial G} u.$$

Harmonic functions on Ω are monotone in Ω . More generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone (see [15]). Our final aim in this note is concerned with growth properties for monotone Sobolev functions in central Herz-Morrey spaces.

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Part I

Isolated Singularities

1 Generalized Riesz potentials

Let us consider the Riesz kernel $I_\alpha(x) = |x|^{\alpha-n}$ of order α and a generalized kernel

$$I_{\alpha,\ell}(x, y) = I_\alpha(x - y) - \sum_{|\lambda| \leq \ell} \frac{(-y)^\lambda}{\lambda!} D^\lambda I_\alpha(x)$$

for an integer ℓ ; when $\ell \leq -1$, $I_{\alpha,\ell}(x, y) = I_\alpha(x - y)$. Note here that $I_{\alpha,\ell}(x, y)$ is Taylor's remainder of $|x + z|^{\alpha-n}$ around $z = 0$.

We define the generalized Riesz potential of order α for a locally integrable function f on the puncture unit ball B_0 by

$$I_{\alpha,\ell}f(x) = \int_{B_0} I_{\alpha,\ell}(x, y) f(y) dy.$$

Here we prepare the estimates for generalized Riesz kernels.

LEMMA 1.1 (cf. [12, Lemma 3.2]). *Let $\ell \geq 0$.*

- (1) $|I_{\alpha,\ell}(x, y)| \leq C|x - y|^{\alpha-n}$ when $|x|/2 < |y| < 2|x|$.
- (2) $|I_{\alpha,\ell}(x, y)| \leq C|y|^{\ell+1}|x|^{\alpha-n-\ell-1}$ when $|y| < |x|/2$.
- (3) $|I_{\alpha,\ell}(x, y)| \leq C|y|^\ell|x|^{\alpha-n-\ell}$ when $2|x| < |y|$.

REMARK 1.2. If u is a superharmonic function on B , then u is represented as the sum of a potential and a harmonic function (see e.g. [2], [3], [14], [23]).

Let u be a superharmonic function on B_0 . In view of Theorems 1.3 and 3.4 in [12], if $r^aS(|u|, r)$ is bounded in $(0, 1)$ for some $a > n - 2$, then

$$\sup_{0 < r < 1/2} r^{a+2-n} \mu(A(0, r)) < \infty \quad (A(0, r) = B(0, 2r) \setminus B(0, r))$$

and u is represented as

$$u(x) = I_{2,\ell}\mu(x) + \text{a harmonic function}$$

near the origin (except at the origin), where $(2 - n + a) - 1 < \ell \leq 2 - n + a$ and $\mu = c(-\Delta)u$ is the Riesz measure; see also [8], [10], [11].

2 Central Herz-Morrey spaces

For $1 \leq p < \infty$ and a real number ν , we consider the family $M^{p,q,\nu}(\mathbf{B})$ of all measurable functions f on \mathbf{B} satisfying

$$\|f\|_{M^{p,q,\nu}(\mathbf{B})} = \left(\int_0^1 (r^\nu \|f\|_{L^p(A(0,r))})^q dr/r \right)^{1/q} < \infty$$

when $0 < q < \infty$ and

$$\|f\|_{M^{p,\infty,\nu}(\mathbf{B})} = \sup_{0 < r < 1} r^\nu \|f\|_{L^p(A(0,r))} < \infty;$$

set $f = 0$ outside \mathbf{B} as before; see e.g. [4], [5], [16].

If $0 < q_1 < q_2 < \infty$, then

$$M^{p,q_1,\nu}(\mathbf{B}) \subset M^{p,q_2,\nu}(\mathbf{B}) \subset M^{p,\infty,\nu}(\mathbf{B})$$

Our space is somewhat a family of functions with finite weighted mixed norm

$$\|f\|_{p,q,\omega} = \left(\int \left(\int |f(x,y)|^p \omega(x,y) dy \right)^{q/p} dx \right)^{1/q} < \infty.$$

3 Sobolev's inequality

Let p^\sharp be the Sobolev exponent of $p > 1$:

$$1/p^\sharp = 1/p - \alpha/n > 0.$$

LEMMA 3.1 (Sobolev's inequality (cf. [1], [23])). *There is a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{p^\sharp}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

Sobolev's inequality is extended to generalize Riesz potentials of functions in central Herz-Morrey spaces.

THEOREM 3.2. *Assume that $\alpha - n/p < \nu < n - n/p$. Then*

$$\|I_\alpha f\|_{M^{p^\sharp,q,\nu}(\mathbf{B})} \leq C \|f\|_{M^{p,q,\nu}(\mathbf{B})}.$$

THEOREM 3.3 (Sobolev's inequality for generalized Riesz potentials). *Assume that $\ell \geq 0$ and $n - n/p + \ell < \nu < n - n/p + \ell + 1$. Then*

$$\|I_{\alpha,\ell} f\|_{M^{p^\sharp,q,\nu}(\mathbf{B})} \leq C \|f\|_{M^{p,q,\nu}(\mathbf{B})}.$$

To prove Sobolev's inequality, for a real number β and $0 < r < 1$, let us consider the Hardy type operators

$$H_\beta^- f(r) = r^{-\beta} \int_{B(0,r)} |y|^{\beta-n} f(y) dy$$

and

$$H_\beta^+ f(r) = r^{-\beta} \int_{\mathbf{B} \setminus B(0,r)} |y|^{\beta-n} f(y) dy$$

for measurable functions f on \mathbf{B} .

LEMMA 3.4. Let $\beta - \nu - n/p > \varepsilon > 0$. Then

$$H_{\beta}^{-} f(r) \leq Cr^{-\varepsilon-n/p-\nu} \left(\int_0^r (t^{\varepsilon+\nu} \|f\|_{L^p(A(0,t))})^q \frac{dt}{t} \right)^{1/q}$$

for all $0 < r < 1$ and $f \in L_{\text{loc}}^1(\mathbb{R}^n)$.

LEMMA 3.5. Let $0 < \varepsilon < -\beta + \nu + n/p$. Then

$$H_{\beta}^{+} f(r) \leq Cr^{\varepsilon-n/p-\nu} \left(\int_{r/2}^1 (t^{-\varepsilon+\nu} \|f\|_{L^p(A(0,t))})^q \frac{dt}{t} \right)^{1/q}$$

for all $0 < r < 1$ and $f \in L_{\text{loc}}^1(\mathbb{R}^n)$.

Proof of Theorem 3.3. Let $\|f\|_{M^{p,q,\nu}(\mathbf{B})} \leq 1$ and $f \geq 0$. For $x \in \mathbf{B}$, set

$$\begin{aligned} I_{\alpha,\ell} f(x) &= \int_{B(0,|x|/2)} I_{\alpha,\ell}(x,y) f(y) dy \\ &\quad + \int_{B(0,2|x|) \setminus B(0,|x|/2)} I_{\alpha,\ell}(x,y) f(y) dy \\ &\quad + \int_{B(0,1) \setminus B(0,2|x|)} I_{\alpha,\ell}(x,y) f(y) dy \\ &= u_1(x) + u_2(x) + u_3(x). \end{aligned}$$

Let $0 < r < 1$. By Lemma 1.1 we have

$$|u_2(x)| \leq C \int_{A(0,r/2) \cup A(0,r)} |x-y|^{\alpha-n} |f(y)| dy$$

for $x \in A(0,r)$, so that Lemma 3.4 gives

$$\|u_2\|_{L^{p^{\sharp}}(A(0,r))} \leq C \|f\|_{L^{p^{\sharp}}(A(0,r/2) \cup A(0,r))}.$$

Hence,

$$\int_0^1 (r^{\nu} \|u_2\|_{L^{p^{\sharp}}(A(0,r))})^q \frac{dr}{r} \leq C \int_0^1 (t^{\nu} \|f\|_{L^{p^{\sharp}}(A(0,r/2) \cup A(0,r))})^q \frac{dt}{t}.$$

By Lemma 1.1 we see that

$$\begin{aligned} |u_1(x)| &\leq C |x|^{\alpha-n-\ell-1} \int_{B(0,|x|/2)} |y|^{\ell+1} f(y) dy \\ &\leq Cr^{\alpha} H_{n+\ell+1}^{-} f(r) \end{aligned}$$

for $x \in A(0,r)$. Hence, using Lemma 3.4, we find

$$\|u_1\|_{L^{p^{\sharp}}(A(0,r))} \leq Cr^{-\varepsilon-\nu} \left(\int_0^r (t^{\varepsilon+\nu} \|f\|_{L^p(A(0,t))})^q \frac{dt}{t} \right)^{1/q}$$

for $0 < \varepsilon < n + \ell + 1 - n/p - \nu$. Consequently,

$$\begin{aligned} \int_0^1 (r^\nu \|u_1\|_{L^p(A(0,r))})^q \frac{dr}{r} &\leq C \int_0^1 \left(r^{-\varepsilon} \int_0^r (t^{\varepsilon+\nu} \|f\|_{L^p(A(0,t))})^q \frac{dt}{t} \right) \frac{dr}{r} \\ &\leq C \int_0^1 (t^{\varepsilon+\nu} \|f\|_{L^p(A(0,t))})^q \left(\int_t^1 r^{-\varepsilon q} \frac{dr}{r} \right) \frac{dt}{t} \\ &\leq C \int_0^1 (t^\nu \|f\|_{L^p(A(0,t))})^q \frac{dt}{t} \end{aligned}$$

Similarly, by Lemma 1.1 we see that

$$\begin{aligned} |u_3(x)| &\leq C|x|^{\alpha-n-\ell} \int_{B \setminus B(0,2|x|)} |y|^\ell f(y) dy \\ &\leq Cr^\alpha H_{n+\ell}^+ f(r) \end{aligned}$$

for $x \in A(0, r)$. Hence, using Lemma 3.5, we find

$$\|u_3\|_{L^p(A(0,r))} \leq Cr^{\varepsilon-\nu} \left(\int_r^1 (t^{-\varepsilon+\nu} \|f\|_{L^p(A(0,t))})^q \frac{dt}{t} \right)^{1/q}$$

for $0 < \varepsilon < -(n + \ell - n/p - \nu)$. Thus,

$$\begin{aligned} \int_0^1 (r^\nu \|u_3\|_{L^p(\cdot)(A(0,r))})^q \frac{dr}{r} &\leq C \int_0^1 \left(r^\varepsilon \int_r^1 (t^{-\varepsilon+\nu} \|f\|_{L^p(A(0,t))})^q \frac{dt}{t} \right) \frac{dr}{r} \\ &\leq C \int_0^1 (t^{-\varepsilon+\nu} \|f\|_{L^p(A(0,t))})^q \left(\int_0^t r^{\varepsilon q} \frac{dr}{r} \right) \frac{dt}{t} \\ &\leq C \int_0^1 (t^\nu \|f\|_{L^p(A(0,t))})^q \frac{dt}{t} \end{aligned}$$

□

4 Growth near the origin of spherical means

The L^q ($1 \leq q < \infty$) means over the spherical surface $S(0, r)$ for a function u is defined by

$$\begin{aligned} S_q(u, r) &= \left(\frac{1}{|S(0, r)|} \int_{S(0, r)} |u(x)|^q dS(x) \right)^{1/q} \\ &= \left(\frac{1}{\omega_{n-1}} \int_{S(0, 1)} |u(r\sigma)|^q dS(\sigma) \right)^{1/q}, \end{aligned}$$

where $S(0, r) = \partial B(0, r)$ and $|S(0, r)| = \omega_{n-1} r^{n-1}$ with ω_{n-1} denoting the area of the unit sphere.

Our aim is to find $d > 0$ such that

$$\liminf_{r \rightarrow 0^+} r^d S_q(I_{\alpha, \ell} f, r) = 0$$

for a function f on B satisfying Herz-Morrey type conditions.

Our result is a continuation of Gardiner's result ([13, 1988]) :

REMARK 4.1. For a Green potential $G\mu$ on B ,

(1) if $(n-1)/(n-2) \leq q < (n-1)/(n-3)$, then

$$\liminf_{r \rightarrow 1} (1-r)^{n-1-(n-1)/q} S_q(G\mu, r) = 0;$$

(2) if $1 \leq q < (n-1)/(n-2)$, then

$$\lim_{r \rightarrow 1} (1-r)^{n-1-(n-1)/q} S_q(G\mu, r) = 0.$$

THEOREM 4.2. Suppose $n - n/p + \ell < \nu < n - n/p + \ell + 1$. If $(n - \alpha p - 1)/(p(n-1)) < 1/q \leq 1/p$, then

$$\liminf_{r \rightarrow 0+} r^{(n-\alpha p+\nu p)/p} S_q(I_{\alpha,\ell} f, r) < \infty$$

for all $f \in M^{p,\nu}(B)$.

THEOREM 4.3. Suppose $n - n/p + \ell < \nu < n - n/p + \ell + 1$. If $(n - \alpha p - 1)/(p(n-1)) < 1/q \leq 1/p$, then

$$\liminf_{r \rightarrow 0+} r^{(n-\alpha p+\nu p)/p} S_q(I_{\alpha,\ell} f, r) = 0$$

for all $f \in M_0^{p,\nu}(B)$.

Part II

Boundary growth properties

5 Superharmonic functions on B

The Riesz kernel is written as

$$|x - y|^{\alpha-n} = \sum_{\ell} (1 - |y|)^{\ell} \phi_{\alpha,\ell}(x, \tilde{y}),$$

where $\tilde{y} = y/|y|$ and

$$\phi_{\alpha,\ell}(x, \tilde{y}) = \sum_{\ell/2 \leq k \leq \ell} a_{\alpha,\ell,k} |x - \tilde{y}|^{\alpha-n-2k} (x \cdot \tilde{y} - 1)^{2k-\ell}.$$

In fact, consider the Taylor expansion of

$$|x - y|^{\alpha-n} = |x - \tilde{y} + t\tilde{y}|^{\alpha-n}$$

and set $t = 1 - |y|$.

Now define

$$K_{\alpha,m}(x, y) = \frac{1}{(n-\alpha)\sigma_n} \begin{cases} |x - y|^{\alpha-n} & (y \in B(0, 1/2)); \\ |x - y|^{\alpha-n} - \sum_{\ell=0}^m (1 - |y|)^{\ell} \phi_{\alpha,\ell}(x, \tilde{y}) & (y \in B \setminus B(0, 1/2)). \end{cases}$$

The following properties for $K_{2,m}$ are fundamental.

LEMMA 5.1 (cf. [9, Lemma 2.2]). (1) $\Delta K_{2,m}(\cdot, y) = \delta_y$ when $n > 2$;

(2) $|K_{\alpha,m}(x, y)| \leq C|x - y|^{\alpha-n-m-1}(1 - |y|)^{m+1}$ when $1 - |y| \leq (\sqrt{2} - 1)|x - \tilde{y}|$.

We show Riesz decomposition for superharmonic functions on \mathbf{B} .

THEOREM 5.2. If u is superharmonic in \mathbf{B} and

$$\liminf_{r \rightarrow 1} (1 - r)^a S(u, r) > -\infty,$$

then

$$u(x) = \int_{\mathbf{B}} K_{2,m}(x, y) d\mu(y) + h_0(x),$$

where h_0 is harmonic in \mathbf{B} and m is an integer greater than a .

Set

$$C(0, r) = B(0, r + (1 - r)/2) \setminus B(0, r - (1 - r)/2)$$

for $0 < r < 1$.

Denote by $\tilde{M}^{p,\nu}(\mathbf{B})$ the family of all functions $f \in L^1_{\text{loc}}(\mathbf{B})$ such that

$$\|f\|_{\tilde{M}^{p,\nu}(\mathbf{B})} = \sup_{0 < r < 1} (1 - r)^\nu \|f\|_{L^p(C(0,r))} < \infty.$$

Now we give a continuation of the results by Gardiner [13].

THEOREM 5.3. Let $1 \leq q < \infty$ and suppose $\|F\|_{\tilde{M}^{p,\nu}(\mathbf{B})} \leq 1$ with $F(y) = (1 - |y|)f(y)$. Then there exists a constant $C > 0$ such that

(1) if $n + m - 1 - \alpha p < (n - 1)/q \leq n + m - \alpha p$, then

$$\liminf_{r \rightarrow 1} (1 - r)^{(n-\alpha p+\nu)/p-(n-1)/q} S_q(K_{\alpha,m}f, r) \leq C;$$

(2) if $n - \alpha p + m < (n - 1)/q < n + m + 1 - \alpha p$, then

$$\sup_{1/2 < r < 1} (1 - r)^{(n-\alpha p+\nu)/p-(n-1)/q} S_q(K_{\alpha,m}f, r) \leq C.$$

6 Isolated singularities for monotone functions in the sense of Lebesgue [18, 1907]

A continuous function u on a domain D is said to be monotone in the sense of Lebesgue [18] if for every subdomain $G, \bar{G} \subset D$,

$$\max_{\bar{G}} u = \max_{\partial G} u \quad \text{and} \quad \min_{\bar{G}} u = \min_{\partial G} u;$$

see Heinonen-Kilpeläinen-Martio [15], Koskela-Manfredi-Villamor [17], Manfredi-Villamor [20, 21], the author [22, 23], the author-Shimomura [24, 25], Villamor-Li [29], Vuorinen [30, 31].

THEOREM 6.1. Suppose $n - 1 < p < \nu + n$. Let u be a function on $\mathbf{B} \setminus \{0\}$ which is monotone in the sense of Lebesgue and satisfies

$$\sup_{0 < r < 1} r^\nu \int_{A(0,r)} |\nabla u(x)|^p dx \leq 1 \quad (p > n - 1).$$

Then

$$\sup_{x \in \mathbf{B}} r^{(n-p-\nu)/p} |u(x)| \leq C < \infty.$$

For monotone functions in sense of Lebesgue, the following is a crucial tool.

LEMMA 6.2 (cf. [20], [21], [23]). If u is monotone in $B(x_0, 2r)$ in the sense of Lebesgue and $p_1 > n - 1$, then $\forall x, y \in B(x_0, r)$

$$|u(x) - u(y)|^{p_1} \leq C r^{p_1 - n} \int_{B(x_0, 2r)} |\nabla u(z)|^{p_1} dz. \quad (6.1)$$

EXAMPLE 6.3. For $\beta > 0$, consider $u(x) = |x|^{-\beta}$. Then

- u is monotone in $\mathbf{B} \setminus \{0\}$ in the sense of Lebesgue ;
- $|\nabla u(x)| \leq C |x|^{-\beta-1}$.

If $-(\beta + 1)p + \nu + n \geq 0$,

$$\sup_{0 < r < 1} r^\nu \int_{A(0,r)} |\nabla u(x)|^p dx < \infty.$$

Hence, letting $\beta = (n - p + \nu)/p \geq 0$, we find

$$\lim_{x \rightarrow 0} |x|^{(n-p+\nu)/p} u(x) = 1.$$

Finally we show boundary growth for monotone functions on \mathbf{B} in the sense of Lebesgue.

THEOREM 6.4. Suppose $n - 1 < p < \nu + n$, $p < q < \infty$. Let u be a function on \mathbf{B} which is monotone in the sense of Lebesgue and satisfies

$$\sup_{0 < r < 1} (1 - r)^\nu \int_{C(0,r)} |\nabla u(x)|^p dx \leq 1.$$

If $(n - 1)/q < (n - p + \nu)/p$, then

$$\sup_{0 < r < 1} (1 - r)^{(n-p-\nu)/p - (n-1)/q} S_q(u, r) \leq C < \infty.$$

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Yoshihiro Mizuta
4-13-11 Hachi-Hom-Matsu-Minami
Higashi-Hiroshima 739-0144, Japan
E-mail : yomizuta@hiroshima-u.ac.jp